Week 13
Nov 28, 2022

2020 A

Two theoretic consequences of Stokes' theorem.

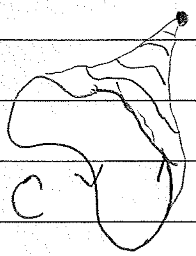
Theorem Let $\vec{F}$ be a smooth v.f in a simply-connected region in $\mathbb{R}^3$. $\vec{F}$ is conservative if it satisfies the component test.

Idea of Pf. Let C be a simply closed curve in the region Ω. It suffices to show

$$\oint_C \vec{F}\cdot d\vec{r} = 0.$$

Since Ω is simply-connected, C can be deformed continuously into a point. The process of deformation forms a surface S whose boundary is C. By Stokes' thm

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,d\sigma.$$



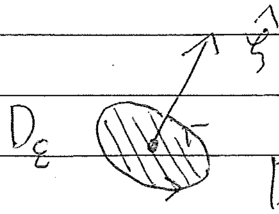
However, by Component Test,

$$\nabla\times\vec{F} = (P_y - N_z)\hat{i} - (P_x - M_z)\hat{j} + (N_x - M_y)\hat{k} = \vec{0}.$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = 0. \quad \#$$

Next, let $\hat{\xi}$ be a direction (ie a unit vector) and D_ε is a disk of radius ε around a point $\vec{P}(x,y,z)$ in the direction $\hat{\xi}$.

The boundary of D_ε, C_ε, is in anticlockwise way w.r.t. $\hat{\xi}$ (so $\hat{\xi}$ is the unit normal for D_ε).



By Stokes' thm,

$$\iint_{D_\varepsilon} \nabla\times\vec{F}\cdot\hat{n}\,d\sigma = \oint_{C_\varepsilon} \vec{F}\cdot d\vec{r} \quad (\text{the circulation of } \vec{F} \text{ around } C_\varepsilon)$$

$$\begin{aligned}\nabla\times\vec{F}\cdot\hat{\xi}\ (\text{at } \vec{p}) &= \nabla\times\vec{F}\cdot\hat{n}\ (\text{at } \vec{p}) \\ &= \lim_{\varepsilon\to 0} \frac{1}{|D_\varepsilon|} \iint_{D_\varepsilon} \nabla\times\vec{F}\cdot\hat{n}\,d\sigma \\ &= \lim_{\varepsilon\to 0} \frac{1}{|D_\varepsilon|} \oint_{C_\varepsilon} \vec{F}\cdot d\vec{r}.\end{aligned}$$

Thus, $\nabla\times\vec{F}\cdot\hat{\xi}$ (at $\vec{p}$) measures how $\vec{F}$ rotates along $\hat{\xi}$ at $\vec{p}$.

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Divergence theorem

The divergence theorem is the 3-dim version of Green's theorem

Recall a version of Green's thm is

$$\iint_D (M_x + N_y)\,dA(x,y) = \oint_C -N\,dx + M\,dy.$$

Now, we have

Divergence theorem Let Ω be a region in $\mathbb{R}^3$ with piecewise smooth boundary Σ. For smooth v.f. $\vec{F}$ in Ω (up to Σ),

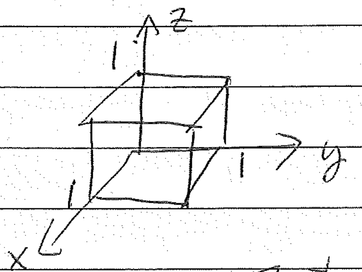
$$\iiint_\Omega \nabla\cdot\vec{F}\,dV(x,y,z) = \oiint_\Sigma \vec{F}\cdot\hat{n}\,d\sigma, \quad \text{where}$$

$\nabla \cdot \vec{F} = M_x + N_y + P_z$ (the divergence of $\vec{F}$)

and $\hat{n}$ is the outward unit normal at Σ.

the proof of Div. thm is similar to that of Green's thm, we omit it

e.g. Find the flux of $\vec{F} = xy\hat{i} + yz\hat{j} + xz\hat{k}$ outward through the surface of the cube at $x=0,1$, $y=0,1$, $z=0,1$.



By Div. thm

$$\text{flux} = \iiint_C \nabla \cdot \vec{F}\, dV.$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$$

$$\text{flux} = \int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz = 3\int_0^1\int_0^1\int_0^1 x\,dx\,dy\,dz \quad \text{(by symmetry)}$$

$$= 3/2 \quad \#$$

When Ω has boundary give by $\Sigma_1, \cdots, \Sigma_n$ (all closed surfaces), div. thm becomes

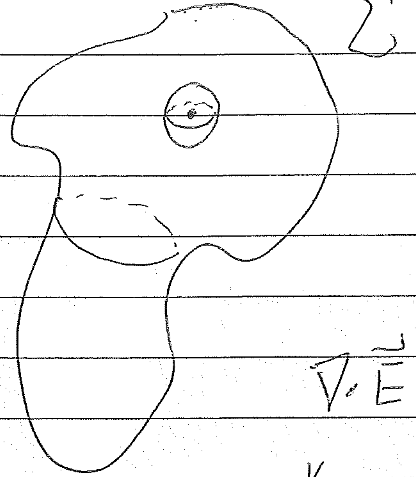
$$\iiint_\Omega \nabla \cdot \vec{F}\, dV = \sum_{j=1}^{n} \iint_{\Sigma_j} \vec{F} \cdot \hat{n}\, d\sigma, \quad \hat{n} \text{ outer unit normal.}$$

As an application, we have

Gauss' law: For any closed surface Σ enclosing a point charge q located at the origin, the out flux of the electric field $\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{\rho^3}$

is always equal to q/ε_0.

Pf:



Place a small sphere S_a (radius a) around the origin. Let Ω_a be the region bounded by Σ and S_a. Then $\vec{E}$ is a smooth v.f. in Ω_a.

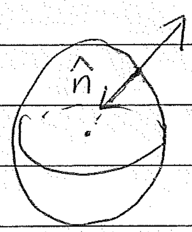
$$\nabla\cdot\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial x}\frac{x}{\rho^3} + \frac{\partial}{\partial y}\frac{y}{\rho^3} + \frac{\partial}{\partial z}\frac{z}{\rho^3}\right]$$

$\rho = (x^2+y^2+z^2)^{1/2}$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\rho^3} - \frac{3x^2}{\rho^5} + \frac{1}{\rho^3} - \frac{3y^2}{\rho^5} + \frac{1}{\rho^3} - \frac{3z^2}{\rho^5}\right]$$

$$= 0$$

$$\therefore \iint_{\Sigma} \vec{E}\cdot\hat{n}\, d\sigma + \iint_{S_a} \vec{E}\cdot\hat{n}\, d\sigma = \iiint_{\Omega_a} \nabla\cdot\vec{E}\, dV = 0$$



$$\iint_{\Sigma} \vec{E}\cdot\hat{n}\, d\sigma = -\iint_{S_a} \vec{E}\cdot\hat{n}\, d\sigma$$

Relative to Ω_a, the outer unit normal at S_a is the inner unit normal of the ball B_a, which is equal to

$$-\frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\iint_{S_a} \vec{E}\cdot\hat{n}\, d\sigma = \iint_{S_a} \frac{q}{4\pi\varepsilon_0}\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{\rho^3}\left(-\frac{1}{a}\right)(x\hat{i}+y\hat{j}+z\hat{k})\, d\sigma$$

$$= \frac{q}{4\pi\varepsilon_0}\iint_{S_a}\left(-\frac{1}{a}\right)\frac{x^2+y^2+z^2}{\rho^3}\, d\sigma \qquad (\rho = a \text{ on } S_a)$$

$$= \frac{q}{4\pi\varepsilon_0}\iint_{S_a} -\frac{1}{a}\frac{a^2}{a^3}\, d\sigma$$

$$= -\frac{q}{4\pi\varepsilon_0}\frac{1}{a^2}\iint_{S_a} d\sigma$$

$$= -\frac{q}{4\pi\varepsilon_0}\frac{1}{a^2}4\pi a^2$$

$$= -\frac{q}{\varepsilon_0}.$$

$$\therefore \iint_{\Sigma} \vec{E}\cdot\hat{n}\, d\sigma = -\iint_{S_a} \vec{E}\cdot\hat{n}\, d\sigma$$

$$= q/\varepsilon_0. \quad \#$$